

POSITIVE DIAGRAMS FOR SEIFERT FIBERED SPACES

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0. Introduction. By a (*Heegaard*) *diagram* we mean a triple $(S; X, Y)$ where S is a closed, connected, orientable surface and X and Y are compact 1-manifolds in S .

A diagram gives rise to a 3-manifold

$$M = S \times [-1, 1] \bigcup_{X \times -1} \text{2-handles} \bigcup_{Y \times 1} \text{2-handles} \bigcup \text{3-handles}$$

obtained by adding 2-handles to $S \times [-1, 1]$ along the curves of $X \times -1$ and $Y \times 1$ and then adding 3-handles along all resulting 2-sphere boundary components. The decomposition of M by $S \times 0$ is the associated (*Heegaard*) *splitting* of M and the genus of S is called the *genus* of the splitting.

A *positive diagram* is a diagram in which S , X , and Y are oriented and the intersection number $\langle X, Y \rangle_p$ of X with Y is $+1$ at each point $p \in X \cap Y$. Every compact, oriented 3-manifold (with no 2-sphere boundary components) is represented by a positive diagram. One can start with an arbitrary diagram for the manifold and eliminate negative crossings by adding trivial handles. The new curves associated with each trivial handle can be oriented so as to introduce only positive crossings. In the process one negative crossing is replaced by three positive crossings and the genus of the associated splitting is increased by one.

Positive diagrams were introduced by Montesinos [M] who observed that they have a nice encoding by the permutations on the $d = \#(X \cap Y)$ intersection points given by flowing along X (respectively Y) from one point to the next, and noted that these permutations are also the monodromy representation for the manifold as a branched cover of S^3 branched over a fixed (universal) branch set. See [H3] for more information.

It is natural to wonder about the relation between the *Heegaard genus*, $hg(M)$, and the *positive Heegaard genus*, $phg(M)$, of M ; where these are defined respectively by minimizing the genus among all splittings of M and among all splittings determined by a positive diagram. From the above it might seem that the difference $phg(M) - hg(M)$ could be arbitrarily large; as one might have to increase the genus by as much as $\#(X \cap Y)/2$ to change all the negative crossings. However this cannot be supported by group theory alone in the following sense. Assuming that at least one of the sides of the splitting is

a handlebody, we get a presentation for $\pi_1(M)$ with generators dual to the attaching curves for the handlebody and relators coming from the attaching curves for the other side. This will be a *positive presentation*: the relators are words in positive powers of the generators.

Fact. *If a group G has a presentation with n generators and m relators, then it has a positive presentation with $n + 1$ generators and $m + 1$ relators.*

Proof. Add a generator x_{n+1} and a relator $x_1 x_2 \dots x_n x_{n+1}$. In every other relator, and for each $i = 1, 2, \dots, n$ replace each occurrence of x_i^{-1} by $x_{i+1} \dots x_n x_1 \dots x_{i-1}$. \square

If one starts with a presentation coming from a Heegaard splitting, the positive presentation obtained as above will not, in general, correspond to a splitting – one cannot represent the new/modified relations by disjoint, embedded simple closed curves in the boundary of a handlebody.

To my knowledge the only class of 3-manifolds (including Heegaard genus > 2) for which there is exact determination of Heegaard genus [BZ],[MS] are the Seifert manifolds (see Section 1 for the notation we use for them). For these we prove:

Theorem A. *Let M be a closed, orientable Seifert fibered space with orientable base space. Then $phg(M) = hg(M)$ except for the following cases:*

(1) $g = 0$; $m \geq 4$ is even; Seifert invariants $= 1/2, 1/2, \dots, 1/2, n/2n + 1$ ($n \geq 1$); $e = m/2$. In these cases $phg(M) \leq hg(M) + 1 = m - 1$, and $phg(M) = hg(M) + 1$ if $m = 4$ or $n > 1$.

(2) $g > 0$; and ($m = 0$; $e = \pm 1$) or ($m = 1$, Seifert invariant $= 1/\alpha$, $e = 0$) or ($m = 1$, Seifert invariant $= (\alpha - 1)/\alpha$, $e = 1$). In these cases $2g + 1 = hg(M) + 1 \leq phg(M) \leq hg(M) + 2$.

(3) $g > 0$; $m \leq 2$; and not in case (2) above. Then $\min\{2g + 1, 2g + m - 1\} = hg(M) \leq phg(M) \leq hg(M) + 1$.

In cases (1) and (2) of Theorem A the minimal genus splittings are “horizontal splittings”. Not many Seifert manifolds have horizontal splittings and they are rarely of minimal genus. The next theorem gives the remaining exceptions.

Theorem B. *Let M be a closed, oriented, Seifert fibered space with orientable base space which is not included in Theorem A and which has a horizontal splitting realizing $hg(M)$. Then M has $g = 0$, $m = 3$, $e = 1$, and Seifert invariants $(1/2, 1/3, n/(6n \pm 1))$, or $(1/2, 1/4, n/(4n \pm 1))$, or $(1/3, 1/3, n/(3n \pm 1))$ for some $n \geq 1$.*

Except for the cases $(1/2, 1/3, 1/5)$, $(1/2, 1/4, 1/3)$, $(1/3, 1/3, 1/2)$, and $(1/2, 1/3, 1/7)$ the horizontal splitting is not represented by a positive diagram.

In the first three exceptions of Theorem B the splitting is also a vertical splitting and so is represented by a positive diagram by Theorem C, below. We are unsure about the fourth case.

Theorem C. *Every vertical splitting of an orientable Seifert fibered space with base space S^2 is represented by a positive diagram.*

The proofs of these theorems depend on the result [MS] that splittings of Seifert manifolds are either “vertical” or “horizontal” and are organized as follows. In section 2 we discuss vertical splittings and prove Theorem C. In section 3 we apply Theorem C and some lifting arguments to establish the upper bounds of Theorem A (Corollary 3.4) and the equality $phg(M) = hg(M)$ when M has at least three singular fibers and the base surface is not S^2 (Theorem 3.6). In section 4 we discuss horizontal splittings and give (Theorem 4.1) the Seifert manifolds for which these horizontal splittings can be of minimal genus. We prove (Theorem 4.2) that, with the indicated exceptions, these minimal genus horizontal splittings are not represented by positive diagrams. This gives the lower bounds of Theorem A and the proof of Theorem B. In section 5 we discuss some of the questions left open.

1. Preliminaries. A *compression body* is a space built as follows. Take a closed, orientable surface S , attach 2-handles to $S \times [0, 1]$ along curves in $S \times 1$, and fill in any resulting 2-sphere boundary components with 3-balls. The image of $S \times 0$ is called the *outer boundary component* of the compression body.

A (*Heegaard*) *splitting* of a compact, orientable 3-manifold M is a representation of M as the union of two compression bodies (handlebodies if M is closed) meeting in a common outer boundary component. Formally, it is a pair (M, S) where $S \subset M$ is a closed, orientable surface which separates M so that the closure of each component of $M - S$ is a compression body with S as outer boundary component.

A (*Heegaard*) *diagram* is a description of a Heegaard splitting by designating the common outer boundary component S and sets of attaching curves in S for the 2-handles each of the compression bodies is to bound. Formally it is a triad $(S; X, Y)$ where S is a closed, orientable surface and X and Y are compact 1-manifolds in S .

Often we will prefer to work with *oriented* objects. So an oriented Heegaard splitting is a pair (M, S) of oriented manifolds (as above) and equivalence of such will be an orientation preserving homeomorphism of pairs. An oriented diagram will be an oriented triad $(S; X, Y)$, and will determine an oriented splitting by the convention that the positive normal to S in M points toward the Y -side of the splitting.

We adopt the following notation and conventions for a closed, oriented Seifert fibered space M :

- $g \geq 0$ will denote the genus of the base surface.
- $m \geq 0$ will denote the number of singular fibers.
- $e \in \mathbb{Z}$ will denote the Euler number.
- $(\alpha_i, \beta_i) \in \mathbb{Z} \times \mathbb{Z}; 1 \leq i \leq m$ will denote the Seifert invariants.

It is understood that α_i and β_i are relatively prime with $0 < \beta_i < \alpha_i$. Some times we will write these as fractions β_i/α_i .

We refer the data as above the *normalized* invariants for M . They are unique to M which is obtained by oriented Dehn filling on a product $F \times S^1$ according to the formula

$$x_i^{\alpha_i} t^{\beta_i} = 1; i = 1, \dots, m$$

$$[a_1, b_1] \dots [a_g, b_g] x_1 \dots x_m t^e = 1$$

where F is an oriented surface with genus g and $m + 1$ boundary components.

We sometimes find it more convenient to work with a non-unique form of the invariants. In particular, finding a positive diagram depends on making a suitable choice of these invariants. So if we take an oriented surface F of genus g and with $r \geq 1$ oriented boundary components x_1, \dots, x_r and we do Dehn filling on $F \times S^1$ according to the formula

$$x_i^{\alpha_i} t^{\beta'_i} = 1; i = 1, \dots, r$$

where $\alpha_i \geq 1$ and β'_i is prime to α_i , the resulting manifold is said to have *non-normalized* invariants $\{g; \beta'_1/\alpha_1, \dots, \beta'_r/\alpha_r\}$.

One can change these invariants by doing twists on vertical annuli in $F \times S^1$. This way one can show

1.1 Proposition. *The normalized invariants for the Seifert fibered space with non-normalized invariants $\{g; \beta'_1/\alpha_1, \dots, \beta'_r/\alpha_r\}$ are*

$$\text{genus} = g$$

$$m = \#\{i : \alpha_i > 1\}$$

Seifert invariants β_i/α_i when $\alpha_i > 1$ and β_i is the least positive residue of β'_i modulo α_i .

$$e = -\sum [\beta'_i/\alpha_i]$$

2. Vertical splittings. It is known [MS] that every splitting of an orientable Seifert manifold with orientable base space is either *horizontal* or *vertical*. All Seifert manifolds have vertical splittings, but most do not admit horizontal splittings. Theorem 0.3 of [MS] describes, in terms of the Seifert invariants, those Seifert manifold which have horizontal splittings. We review these definitions here.

Let M be a Seifert manifold with base surface B and projection $f : M \rightarrow B$. Suppose we have a cell decomposition of B such that $B = D \cup E \cup F$ where each of D, E, F is a disjoint union of closed 2-cells of the decomposition, each component of D and of E contains at most one singular point, which is an interior point, each component of F is a square containing no singular point and having one pair of opposite sides in D and the other pair in E , $\text{Int}(D) \cap \text{Int}(E) = \text{Int}(D) \cap \text{Int}(F) = \text{Int}(E) \cap \text{Int}(F) = \emptyset$, and $D \cup F$ and $E \cup F$ are connected. See Figure 1.

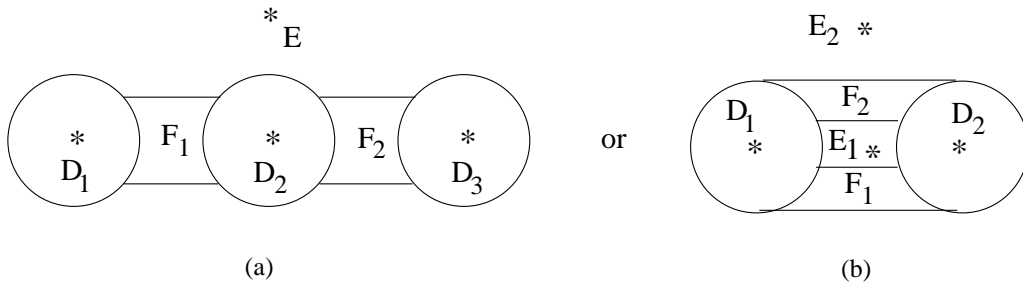


FIGURE 1

Then $f^{-1}(D)$ is homeomorphic to $D \times S^1$, the same holds for E and F , and we consistently fix such identifications. Let $V_1 = D \times S^1 \cup F \times [0, 1/2]$ and $V_2 = E \times S^1 \cup F \times [1/2, 1]$; where $S^1 = [0, 1]/0 \sim 1$. Put $S = V_1 \cap V_2 = \partial(V_1) = \partial(V_2)$. Then (M, S) is a splitting of M of genus

$$\begin{aligned} g &= \beta_0(D) + \beta_1(D \cup F) = \beta_0(E) + \beta_1(E \cup F) = 1 + \beta_0(F) \\ &= 2g(S) + \beta_0(D) + \beta_0(E) - 1 \end{aligned}$$

which we call a *vertical splitting*. These calculations immediately give

2.1 Proposition. *A vertical splitting of a closed, orientable Seifert manifold over an orientable surface of genus g and with m singular fibers has genus at least*

$$\max\{2g + 1, 2g + m - 1\}$$

Proof of Theorem C. We may assume $m \geq 3$. We take the vertical splitting as described above where D has components D_1, \dots, D_{m_1} and E has components E_1, \dots, E_{m_2} ; where $m_1 + m_2 = m$. Let μ_i and ν_j be the positively oriented boundaries of D_i and E_j respectively. M is obtained from $(\partial D \cup \partial E \cup F) \times S^1$ by attaching solid tori according to formula

$$\begin{aligned} \mu_i^{\alpha_i} t_i^{\beta_i} &= 1; \quad i = 1, \dots, m_1 \\ \nu_j^{\alpha_{m_1+j}} s_j^{\beta_{m_1+j}} &= 1; \quad j = 1, \dots, m_2 \end{aligned}$$

corresponding to some non-normalized coordinates $\{\beta_i/\alpha_i\}$. The curves t_i (s_j) are positively oriented vertical curves in $f^{-1}(D_i \cap E)$ ($f^{-1}(E_j \cap D)$). Recall that, by 2.1, we may change the β_i modulo α_i as long as we don't change $\sum[\beta_i/\alpha_i]$. We will impose more conditions on these invariants later.

Let Γ_D (respectively Γ_E) be the graph whose vertices correspond to the components of D (respectively of E) and whose edges correspond to the components of F . There are natural embeddings $\Gamma_D \subset D \cup F, \Gamma_E \subset E \cup F$.

The meridian disks for V_1 will be:

- (i) meridian disks for the filling solid $f^{-1}(D_i)$; together with
- (ii) vertical disks $A_p \subset F_p \times [0, 1/2]$ corresponding to those components F_p of F not in a fixed maximal tree of Γ_D . These will be of the form $c_p \times [0, 1/2]$, where c_p is an arc in F_p separating its edges lying in D .

Similarly the meridian disks for V_2 will be meridian disks for the $f^{-1}(E_j)$ and vertical disks $B_q \subset F_q \times [1/2, 1]$ corresponding to components F_q of F not in a maximal tree in Γ_E .

We need the following conditions.

- (1) Each A_p has vertical sides in some $f^{-1}(E_{j_1})$ and some $f^{-1}(E_{j_2})$. We require that $\beta_{m_1+j_1}$ and $\beta_{m_1+j_2}$ have opposite sign. Same for the β 's on opposite sides of each B_q .
- (2) $A_p \cap B_q = \emptyset$ for all p, q .
- (3) s_j is to be chosen in $f^{-1}(D_i \cap E_j)$ with $\beta_i > 0$. Similarly t_i is to be chosen in $f^{-1}(D_i \cap E_j)$ with $\beta_{m_1+j} > 0$

First we show that these conditions will produce a positive diagram. Their justification will be given afterwards. Figure 2 illustrates this construction using the decomposition of Figure 1b.

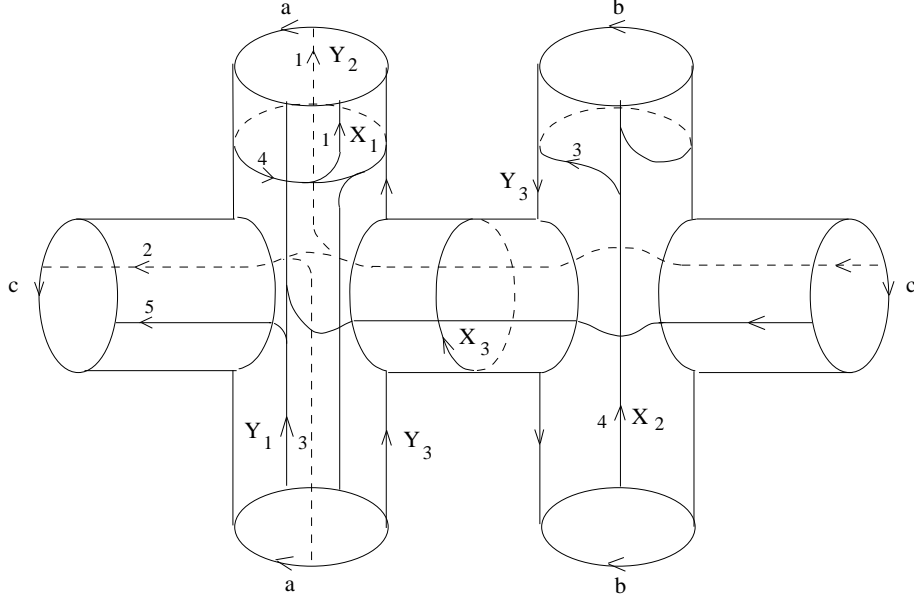


FIGURE 2. Positive diagram for Seifert manifold ($g = 0; 1/4, 2/3, 3/5, 1/2; e = 5$); for which we use non-normalized invariants $(1/4, -4/3, 3/5, -5/2)$. Curves are represented as weighted train tracks.

The meridian curve X_i for $f^{-1}(D_i)$ is chosen in a neighborhood of $t_i \cup \mu_i \times 3/4$ and oriented so that it is homotopic in $f^{-1}(\partial D_i)$ to $\mu_i^{\alpha_i} t_i^{\beta_i}$ or $\mu_i^{-\alpha_i} t_i^{-\beta_i}$ according as $\beta_i > 0$ or $\beta_i < 0$. We similarly choose the meridian Y_j of $f^{-1}(E_j)$.

Now we may suppose $S = \partial V_1 = \partial V_2$ is oriented so that $\langle \mu_i, t \rangle = +1$ for every positively oriented vertical curve t in $f^{-1}(\partial D_i)$. It follows that $\langle t, \nu_j \rangle = +1$ for every positively oriented vertical curve t in $f^{-1}(\partial E_j)$. Now $X_i \cap Y_j$ lies in a neighborhood of $((\mu_i \times 3/4) \cap s_j) \cup (t_i \cap (\nu_j \times 1/4))$. The first (second) of these terms is nonempty only if $s_j(t_i) \subset f^{-1}(D_i \cap E_j)$. Condition (3) then assures us that $\langle X_i, Y_j \rangle = +1$ at each crossing point.

By (2) a vertical A_p will only meet the two Y curves Y_{j_1} and Y_{j_2} . By condition (1) ∂A_p can be oriented to meet both of these positively at each point. Similarly each ∂B_q can be oriented so that X crosses it positively at every point.

To justify condition (1), let T be the subgraph of Γ_E dual to the edges of Γ_D not in a maximal tree T_D in Γ_D . T is connected as $S^2 - T_D$ is connected. T must be a tree; otherwise some loop in T meets a loop in Γ_D in a single point. This is impossible in S^2 . Thus we can put the vertices of T into two classes according to whether the simplicial distance in T to a fixed vertex is even or odd. Each edge of T joins a vertex of one class to one of the other class. We can then choose the β 's corresponding to one class positive and to the other negative. This is possible as we can change the β 's (modulo the corresponding α 's) as long as we keep $\sum [\beta_i / \alpha_i]$ constant.

Note that the above choices of $T_D \subset \Gamma_D$ and $T \subset \Gamma_E$ (which is clearly maximal) satisfy (2); since $(\Gamma_D - T_D) \cap (\Gamma_E - T) = \emptyset$

To get (3) it suffices to show that each E_j meets some D_i with $\beta_i > 0$ (and vice versa). Note that E_j is contained in a unique component E'_j of $S^2 - \Gamma_D$. If E_j only met D_i 's with $\beta_i < 0$, then, by (1), all the edges of Γ_D in $\partial E'_j$ lie in the maximal tree. This is impossible unless Γ_D is itself a tree (and E has a single component). This last possibility is easily handled.

3. Lifting to covers. The following result comes from lifting a positive diagram for the base.

3.1 Lemma. *If M is represented by a positive diagram of genus g and $p : \tilde{M} \rightarrow M$ is a λ -sheeted covering space, then \tilde{M} is represented by a positive diagram of genus $\tilde{g} = \lambda(g - 1) + 1$.*

The next result describes how to lift a Seifert fibration to a particular kind of cover.

3.2 Lemma. *Let M be an orientable Seifert fibered space over an orientable surface of genus g and with non-normalized invariants $\beta_1/\alpha_1, \dots, \beta_r/\alpha_r$; $r > 0$. So M is obtained by Dehn filling on $F \times S^1$ where F is an oriented surface of genus g and with oriented boundary components x_1, \dots, x_r . Let $p : \tilde{F} \rightarrow F$ be a λ -sheeted covering so that for each $i = 1, \dots, r$ $p^{-1}(x_i)$ has components $\tilde{x}_{i,1}, \dots, \tilde{x}_{i,r_i}$ with $p|_{\tilde{x}_{i,j}} : \tilde{x}_{i,j} \rightarrow x_i$ a $b_{i,j}$ -sheeted cover and with $b_{i,j}$ dividing α_i . Put $\alpha_{i,j} = \alpha_i/b_{i,j}$.*

Then $p \times 1 : \tilde{F} \times S^1 \rightarrow F \times S^1$ extends to a λ -sheeted covering space $\tilde{M} \rightarrow M$ where \tilde{M} is the orientable Seifert fibered space over the orientable surface of genus

$$\tilde{g} = \lambda(g - 1) + 1 + (r\lambda - \sum r_i)/2 = \lambda(g - 1) + 1 + \sum (b_{i,j} - 1)/2$$

with non-normalized invariants $\{\beta_i/\alpha_{i,j}; \quad i = 1, \dots, r, j = 1, \dots, r_i\}$.

3.3 Theorem. *Let \tilde{M} be an orientable Seifert fibered space over an orientable surface of genus g with at most three singular fibers. Then \tilde{M} is a $(2g + 1)$ -sheeted cover of an orientable Seifert space over S^2 with three singular fibers.*

Proof. We represent \tilde{M} with non-normalized invariants β_i/α_i ; $i = 1, 2, 3$ (some $\alpha_i = 1$ if $m < 3$). By Lemma 3.5 below, with $\lambda = 2g + 1$, we can find integers β_i^* with $\beta_i^* \equiv \beta_i \pmod{\alpha_i}$, $(\lambda, \beta_i^*) = 1$, and $[\beta_1^*/\alpha_1] + [\beta_2^*/\alpha_2] + [\beta_3^*/\alpha_3] = [\beta_1/\alpha_1] + [\beta_2/\alpha_2] + [\beta_3/\alpha_3]$.

Let F be a sphere with three holes. Since λ is odd, there is a λ -sheeted cyclic covering space $p : \tilde{F} \rightarrow F$ such that the inverse image of each boundary component of F is connected.

Now apply Lemma 3.2 to the Dehn filling of $F \times S^1$ with non-normalized invariants $\beta_i^*/\lambda\alpha_i$; $i = 1, 2, 3$. The λ -sheeted cover of M given by 3.2 is our given \tilde{M} . \square

By Theorem C an orientable Seifert fibered space over S^2 with three singular fibers is represented by a positive diagram of genus two. Applying Lemma 3.1 and Theorem 3.3 to this gives

3.4 Corollary. *An orientable Seifert fibered space over an orientable surface of genus g with at most three singular fibers is represented by a positive diagram of genus $2g + 2$.*

3.5 Lemma. *Given $(\alpha_i, \beta_i) \in \mathbb{Z} \times \mathbb{Z}$ with $\alpha_i \geq 1$ and $(\alpha_i, \beta_i) = 1$; $i = 1, \dots, n$, and given odd $\lambda \in \mathbb{Z}$ there exist $\beta_i^* \in \mathbb{Z}$; $i = 1, \dots, n$ satisfying*

- (1) $\beta_i^* \equiv \beta_i \pmod{\alpha_i}$
- (2) $(\beta_i^*, \lambda) = 1$
- (3) $\sum[\beta_i^*/\alpha_i] = \sum[\beta_i/\alpha_i]$

Proof. We induct on the number of distinct prime factors of λ .

Let $\lambda = p^r$ where p is an odd prime. If $(p, \beta_i) = 1$ for all i , we are already done. So suppose $p|\beta_i$ if and only if $1 \leq i \leq m$. Then $(p, \alpha_i) = 1$; $1 \leq i \leq m$; so $(p, \beta_i + k\alpha_i) = 1$; $1 \leq i \leq m$ if $1 \leq |k| < p$.

If $m = 1$, then p cannot divide both $\beta_2 + \alpha_2$ and $\beta_2 - \alpha_2$; otherwise p divides $2\beta_2$; hence p divides β_2 . So, say, $(p, \beta_2 - \alpha_2) = 1$. Put $\beta_1^* = \beta_1 + \alpha_1$, $\beta_2^* = \beta_2 - \alpha_2$, and $\beta_i^* = \beta_i$; $i \geq 3$.

If m is even, put $\beta_i^* = \beta_i + \alpha_i$; $1 \leq i \leq m/2$, $\beta_i^* = \beta_i - \alpha_i$; $m/2 < i \leq m$, and $\beta_i^* = \beta_i$; $m < i$.

If $m > 1$ is odd, put $\beta_1^* = \beta_1 + 2\alpha_1$, $\beta_i^* = \beta_i + \alpha_i$; $2 \leq i \leq (m-1)/2$, $\beta_i^* = \beta_i - \alpha_i$; $(m-1)/2 < i \leq m$, and $\beta_i^* = \beta_i$; $m < i$.

In each case $\{\beta_i^*\}$ satisfy (1), (2), and (3).

Now suppose $\lambda = \lambda'\lambda''$ where $(\lambda', \lambda'') = 1$, and that we have $\{\beta_i'\}$ and $\{\beta_i''\}$ satisfying (1), (2), and (3) relative to λ' and λ'' respectively.

Since $\alpha_i = (\alpha_i\lambda', \alpha_i\lambda'')$ divides $\beta_i' - \beta_i''$ we can use the Chinese remainder theorem to find β_i^* satisfying $\beta_i^* \equiv \beta_i' \pmod{\alpha_i\lambda'}$ and $\beta_i^* \equiv \beta_i'' \pmod{\alpha_i\lambda''}$.

Then the $\{\beta_i^*\}$ satisfy (1), (2), and (3') $\sum[\beta_i^*/\alpha_i] = \sum[\beta_i/\alpha_i] + k\lambda$, for some k . Put $\beta_1^{**} = \beta_1^* - k\alpha_1\lambda$ and $\beta_i^{**} = \beta_i^*$; $i \geq 2$. The $\{\beta_i^{**}\}$ satisfy (1), (2), and (3). \square

3.6 Theorem. *Let M be a closed oriented Seifert fibered space over an orientable surface of genus $g > 0$ and with $m \geq 3$ singular fibers. Then $phg(M) = hg(M) = 2g + m - 1$.*

Proof. The fact that $hg(M) = 2g + m - 1$ is established in [BZ]. Note that with these assumptions the minimal genus splitting is always a vertical splitting. The case $m = 3$ follows directly from Corollary 3.4.

So we assume that $m > 3$ and that M is defined by non-normalized invariants β_i/α_i ; $i = 1, \dots, m$ with $\beta_i < 0$ for $i > 3$.

Let M' be the Seifert fibered space over the surface of genus g with three singular fibers determined by the non-normalized invariants β_i/α_i ; $i = 1, 2, 3$.

By 3.4 M' is represented by a positive diagram $(S; X, Y)$ of genus $2g + 2$. From the proof of 3.4 (and its predecessors) there is a positively oriented regular fiber $t \subset S$ so that $\langle X, t \rangle_p = +1$ ($\langle Y, t \rangle_q = -1$) for each point $p \in X \cap t$ ($q \in Y \cap t$), and so that there is an interval $J \subset S$ with $t \cap Y \subset J$ and $J \cap X = \emptyset$.

Choose parallel copies t_4, \dots, t_m in S which meet X and Y as t does and push these to the Y side, V_2 , of the splitting by disjoint annuli $t_i \times [0, 1]$, where $t_i = t_i \times 0$.

Now M is obtained from M' by surgery on the curves $t_i \times 1$; $i \geq 4$; so “small” regular neighborhoods of these curves are replaced by solid tori W_i ; $i \geq 4$. We tube these to V_1 along regular neighborhoods of the $J_i \times [0, 1]$ to obtain a handlebody V_1^* which will be

half of the splitting for M . The other half, V_2 , is homeomorphic to the result of removing an open regular neighborhood of $\bigcup_{i \geq 4} (J_i \times [0, 1] \cup t_i \times 1)$.

We get a diagram for this splitting as follows. We add to X meridian curves for the W_i . These are oriented as $\mu_i^{-\alpha_i} t_i^{-\beta_i}$ for a positively oriented regular fiber t_i and transversal μ_i in ∂W_i .

We modify the curves of Y near where they cross the J_i so as to run around the other edge of a regular neighborhood of $J_i \times [0, 1]$. See Figure 3. To these we add the curves $((t_i - J_i) \times [0, 1]) \cap \partial V_2^*$; $i \geq 4$.

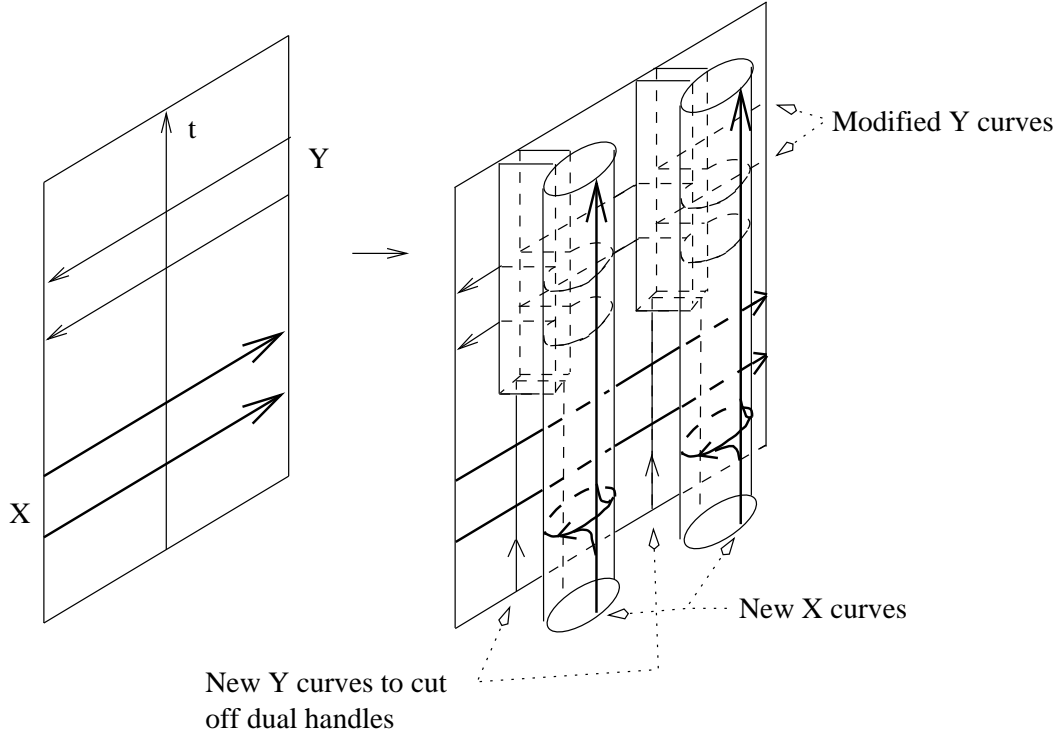


FIGURE 3. Modifying a positive diagram to accommodate more singular fibers.

One verifies that this construction can be made so that all new intersections will be positive. \square

4. Horizontal splittings. Let $F \neq B^2$ be a compact, connected, orientable surface with one boundary component. Consider a surface bundle over S^1

$$N = F \times [0, 1] / (x, 0) \sim (\phi(x), 1)$$

where $\phi : F \rightarrow F$ is an orientation preserving homeomorphism with $\phi|_{\partial(F)} = 1$. Note that $\lambda = \partial(F) \times 0$ and $\mu = x_0 \times [0, 1] / \sim$ ($x_0 \in \partial(F)$) form a basis for $H_1(\partial(N))$.

Let

$$M = N \cup_h B^2 \times S^2$$

be a Dehn filling of N where $h : \partial(B^2 \times S^1) \rightarrow \partial(N)$ is a homeomorphism such that $h(\partial(B^2) \times 0)$ is homologous to $\mu + n\lambda$ for some $n \in \mathbb{Z}$.

Now $h^{-1}(\partial(F) \times \{0, 1/2\})$ bounds an annulus $A \subset B^2 \times S^1$ which splits $B^2 \times S^1$ into two solid tori U_1 and U_2 with (U_i, A) homeomorphic to $(I \times I \times S^1, I \times 0 \times S^1)$.

Then

$$V_1 = F \times [0, 1/2] \cup_h U_1 \text{ and } V_2 = F \times [1/2, 1] \cup_h U_2$$

are handlebodies of genus $2g(F)$, and thus give a splitting of M which we call a *horizontal splitting*.

If ϕ is (isotopic to) a periodic homeomorphism, then N is Seifert fibered, and if $n \neq 0$ this extends to a Seifert fibration of M – with a singular fiber in $M - N$ if $n \neq \pm 1$. The other singular fibers correspond to fixed points of some power ϕ^k of ϕ where k is some proper divisor of the period p of ϕ . The base surface of this Seifert fibration will be \hat{F}/ϕ where \hat{F} is obtained from F by capping off $\partial(F)$ with a 2-cell (on which $\phi = 1$).

Using the Riemann–Hurwitz formula and 2.1 one can show that the genus of this horizontal splitting is less than the genus of any vertical splitting only when $\hat{F}/\phi = S^2$ and $p = 2$ or there is at most one singular fiber (and $\hat{F} \neq S^2$). Moreover, the only additional cases with “horizontal genus” = “vertical genus” occur with $F = T^2$. With a bit more analysis, including the classification of periodic homeomorphisms of T^2 [H1; Theorem 12.11], one can establish

4.1 Theorem. *Let M be a closed oriented Seifert fibered space over an oriented surface S which has a horizontal splitting of genus g_{hor} . Let g_{ver} be the minimal genus of a vertical splitting of M . Then*

If $g_{hor} < g_{ver}$, then either

(1.1) M is Seifert fibered over S^2 with an even number $m \geq 4$ of singular fibers, Seifert invariants $1/2, 1/2, \dots, 1/2, n/(2n+1)$; $n \geq 1$, and Euler number $e = m/2$, or

(1.2) M is Seifert fibered over a surface of genus $g > 0$ with at most one singular fiber and non-normalized invariant $\pm 1/n$; $n \geq 1$.

If $g_{hor} = g_{ver}$, then M is Seifert fibered over S^2 with $m = 3$ singular fibers, Euler number $e = 1$ and invariants either

(2.1) $1/2, 1/3, n/(6n \pm 1)$; $n \geq 0$, or

(2.2) $1/2, 1/4, n/4n \pm 1$; $n \geq 0$, or

(2.3) $1/3, 1/3, n/(3n \pm 1)$; $n \geq 0$.

In most of these cases we can show that the horizontal splitting of M is not represented by a positive diagram:

4.2 Theorem. *Let M satisfy the conclusion of 4.1. Then the minimal genus horizontal splitting of M is not represented by a positive diagram in the cases:*

(1.1) provided $n \geq 2$ or $m = 4$,

(1.2) all cases, and

(2) all cases but $(1/2, 1/3, 1/5)$, $(1/2, 1/4, 1/3)$, $(1/3, 1/3, 1/2)$, and $(1/2, 1/3, 1/7)$.

Proof. First, the easiest case: (1.2). Here the horizontal splitting has genus $2g$ which is also the rank of $H_1(M)$. A positive diagram for this splitting would give a positive presentation

for $\pi_1(M)$ with $2g$ generators and $2g$ relations and thus a $2g \times 2g$ presentation matrix for \mathbb{Z}^{2g} with all non-negative, and some positive entries. This is impossible.

Next we consider case (1.1). We note that identifying opposite edges on a regular $2k$ -gon P (reversing orientation) produces an orientable surface \hat{F} of genus $g = k/2$ if k is even or $g = (k-1)/2$ if k is odd. Rotation of P by 180 deg induces an orientation preserving involution $\phi : \hat{F} \rightarrow \hat{F}$. Note that the fixed points of ϕ come from the mid points of the edges of P , the center c of P , and, in case k is even, the vertices of P . So the number of singular fibers is $m = 2g + 2$.

Let F be obtained from \hat{F} by removing an invariant neighborhood of c . Then the manifold M of (1.1) is obtained by Dehn filling on the bundle $F \times [0, 1]/(x, 0) \sim (\phi(x), 1)$. Specifically, let $\lambda = \partial F \times 0$, as oriented by F and let μ be a transversal to λ which traverses once in the positive t direction while traversing one-half turn in the positive λ direction (from $x_0 \in \partial F$ to $\phi(x_0)$). Then M is obtained by the filling corresponding to $\mu\lambda^n = 1$; $n \in \mathbb{Z} - 0$.

Note that reflection through a diameter of P induces an involution of $F \times [0, 1]/\phi$ which takes λ to λ^{-1} and μ to $\mu\lambda^{-1}$. Thus the surgeries $\mu\lambda^n = 1$ and $\mu\lambda^{-n-1} = 1$ produce the same manifold (in case the reader wonders if I have forgotten half of them). So we assume that $n \geq 1$.

The horizontal splitting gives a representation

$$M = F \times [0, 1] \cup_g F \times [0, 1]$$

where

$$g : \partial(F \times [0, 1]) \rightarrow \partial(F \times [0, 1])$$

is a level preserving homeomorphism such that $g(x, 0) = (x, 0)$, $g(x, 1) = (\phi(x), 1)$ for all $x \in F$, and $g|_{\partial(F) \times [0, 1]}$ is a $n + 1/2$ twist. We get a diagram $(S; X, Y)$ for this splitting as follows. We put $S = \partial(F \times [0, 1])$. We take arcs a_1, \dots, a_{2g} which cut F to a 2-cell as shown in Figure 4. Then $X = \bigcup_i \partial(a_i \times [0, 1])$ and $Y = \bigcup_i g(\partial(a_i \times [0, 1]))$.

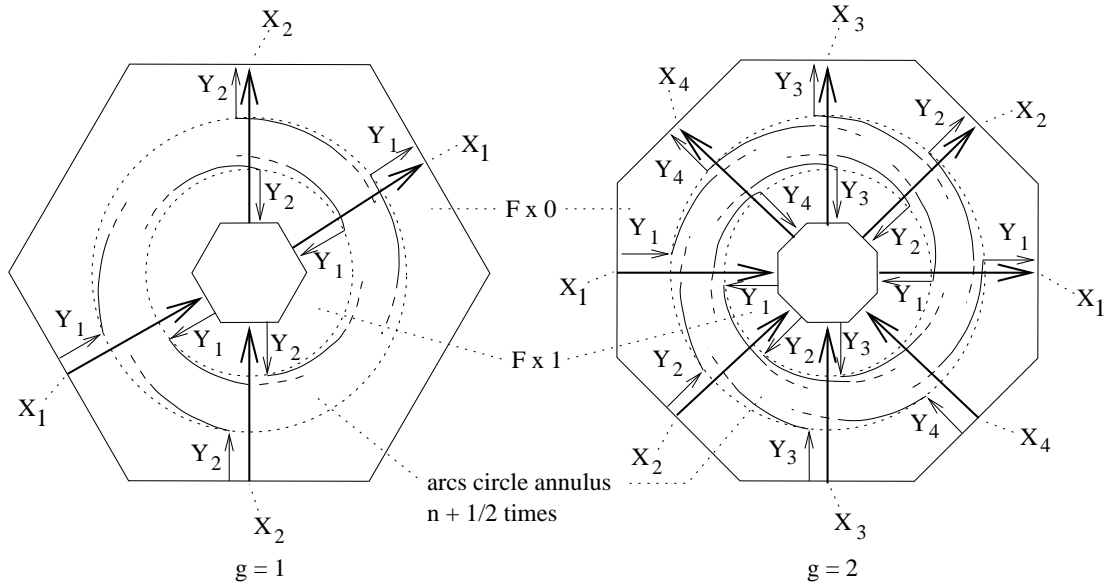


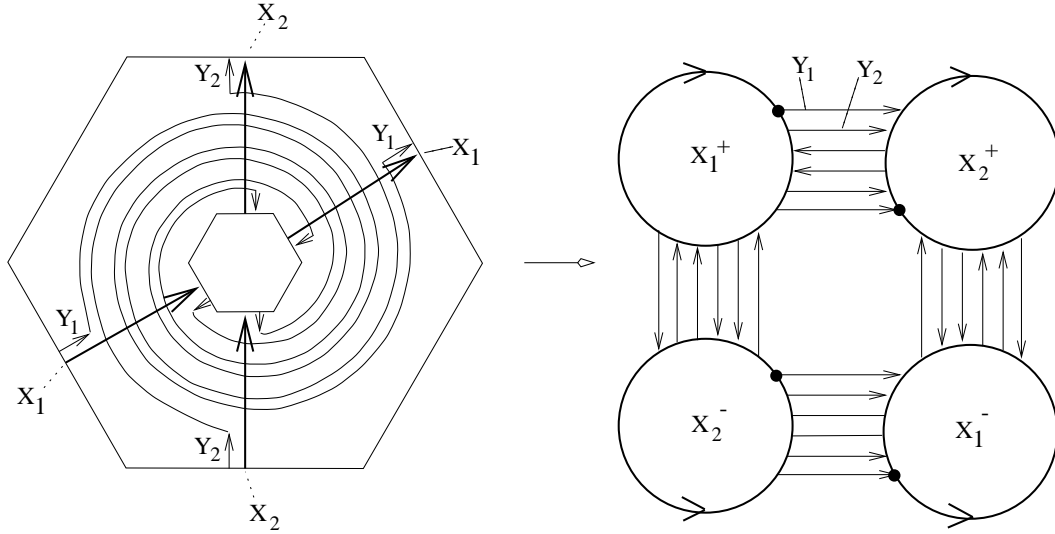
FIGURE 4. Horizontal splitting for $(g = 0; 1/2, 1/2, \dots, 1/2, n/2n+1; e = m/2)$: $1/n$ surgery on the S_g bundle, $g = m/2-1$, over S^1 with period two monodromy.

Now most of the components of $S - (X \cup Y)$ are squares with two opposite sides in X and the other two sides in Y . We form X -stacks by taking maximal unions of such squares along common X -edges, and similarly we form Y -stacks. See [H2] for more details about stacks and their use in analysing Heegaard splittings.

Now suppose $(S; X^*, Y^*)$ is a positive diagram for the splitting – with X^* meeting X minimally, etc. Note that $(S; X, Y)$ has no waves. As described in [H2] each component of X^* must contain at least two Y -stack crossings and each component of Y^* must contain at least two X -stack crossings.

Now suppose that $n \geq 2$. Then if a is any X -stack crossing and b any Y -stack crossing, $a \cup b$ separates S into two components each of which is incompressible in $F \times [0, 1]$. Then any component of X^* other than the one containing b must cross a twice in opposite directions. Thus $(S; X^*, Y^*)$ is not positive.

Now suppose that $n = 1$ and $m = 4$. Then $g = 1$. See Figure 5. Let V_X (respectively V_Y) denote the handlebodies of the splitting with X (respectively Y) as meridians. Suppose $D \subset V_X$ and $E \subset V_Y$ are properly embedded, essential, oriented disks such that $\langle \partial(D), \partial(E) \rangle_p = +1$ at every point $p \in \partial(D) \cap \partial(E)$. We may suppose that X meets $\partial(D)$ and Y meets $\partial(E)$ minimally.

FIGURE 5. Horizontal splitting for $(g = 0; 1/2, 1/2, 1/2, 1/3; e = 2)$.

Claim. $\partial(D) \cap \partial(E)$ contains crossings of two distinct Y -stacks (X -stacks).

We complete the proof of this case modulo the claim – whose proof is then given. So let $a_1, a_2 \subset \partial(D)$ ($b_1, b_2 \subset \partial(E)$) be the Y -stack (X -stack) crossings from the claim. Because they are crossings of different stacks, we see that $J = a_1 \cup a_2 \cup b_1 \cup b_2$ separates S into two incompressible components.

Now suppose that D and E come from a positive diagram for the given splitting. There is a second disk $E' \subset V_Y$ coming from the diagram. Each a_i crosses b_1 and b_2 in the same direction; so a_1 and a_2 are consistently oriented in J . Now $\partial(E')$ must cross J at least two times in opposite directions. Since $E' \cap (b_1 \cup b_2) = \emptyset$, $\partial(E')$ must cross $a_1 \cup a_2$ at least two times in opposite directions. So the diagram was not positive after all.

Proof of Claim. The argument is symmetric in X and Y . Let $D_1, D_2 \subset V_X$ be the disks bounded by the components of X . Cutting V_X along $D_1 \cup D_2$ produces a 3-cell B whose boundary contains two copies D_{i+} and D_{i-} of D_i ; $i = 1, 2$. We get a graph $\Gamma \subset \partial(B)$ whose “fat” vertices are these $D_{i\pm}$ and whose edges are the arcs of $\partial(D)$ cut open.

Now Γ has at least two loops – coming from outermost components of $D - (D_1 \cup D_2)$. Suppose that one of these loops, a is based at, say, D_{1+} . This loop must separate one (*the singleton*) of the remaining vertices from the other two. If the singleton is D_{1-} , then a must cross both of the Y -stacks with one side in D_{1-} , and we are done.

So, say, the singleton is D_{2+} . Let $n_{i,j}$; $i, j \in \{1^\pm, 2^\pm\}$ denote the number of edges of Γ with one vertex in D_i and the other in D_j . So $n_{1+,1+} > 0$, and $n_{1-,2+} = n_{2-,2+} = n_{2+,2+} = 0$.

The edges of Γ must match up on reidentification. This forces consistency equations:

$$n_{1-,1-} + n_{1-,2-} + n_{1-,2+} = n_{1+,1+} + n_{1+,2-} + n_{1+,2+},$$

$$n_{1-,2-} + n_{1+,2-} + n_{2-,2-} = n_{1-,2+} + n_{1+,2+} + n_{2+,2+}$$

If $n_{1-,1-} = 0$, then putting in all of the 0's and solving for $n_{1-,2-}$ from the first equation and putting this in the second gives

$$n_{1+,1+} + 2n_{1+,2-} + n_{2-,2-} = 0$$

This is impossible, as all terms are non-negative and $n_{1+,1+} > 0$. So $n_{1-,1-} > 0$. This forces the singleton to be D_{1-} and completes the proof of the claim.

Finally we consider the case (2). Here M is a torus bundle over S^1 (see [H1; Theorem 12.11]), and is obtained as above with ϕ the rotation of the hexagonal torus by 60° or 120° or rotation of the square torus by 90° (the period 2 rotation of either gives rise to four singular fibers and is included in case (1.2) above). Figure 6 shows diagrams for the corresponding splittings. The right hand side shows the diagrams in a more standard form and are valid for $n > 0$. With notation as in (1.2) the manifolds come from $\mu\lambda^k = 1$ Dehn filling on $F \times [0, 1]/\phi$ for some $k \in \mathbb{Z} - 0$. Here we do not have duality between positive and negative k . The n in the statement is $n = |k|$ and the sign in the denominator of the third invariant is $+$ or $-$ according as $k > 0$ or $k < 0$.

FIGURE 6. Horizontal splittings for the $1/\pm n$ surgeries on the torus bundles over S^1 with periodic monodromy ($\neq 2$).

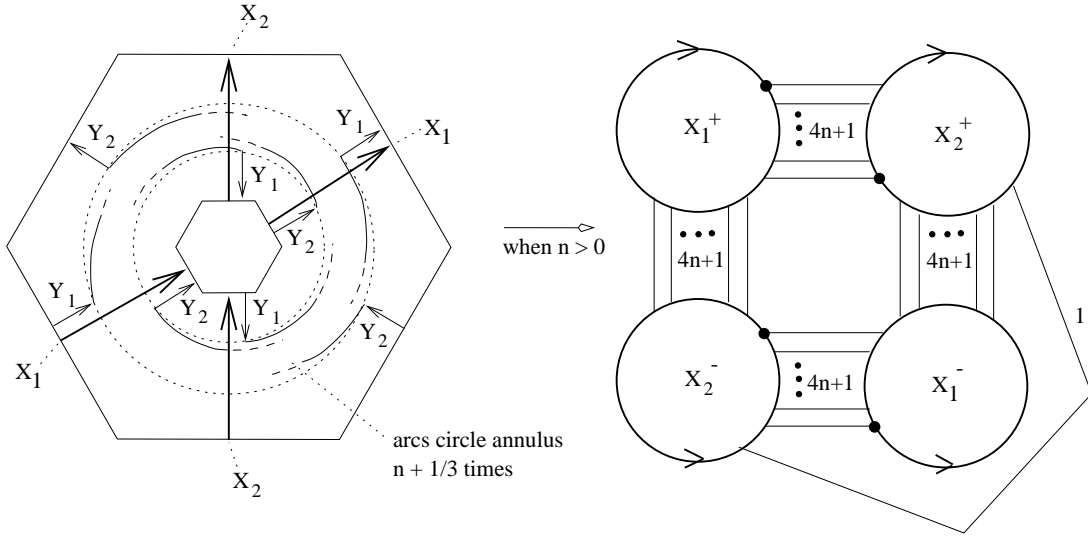
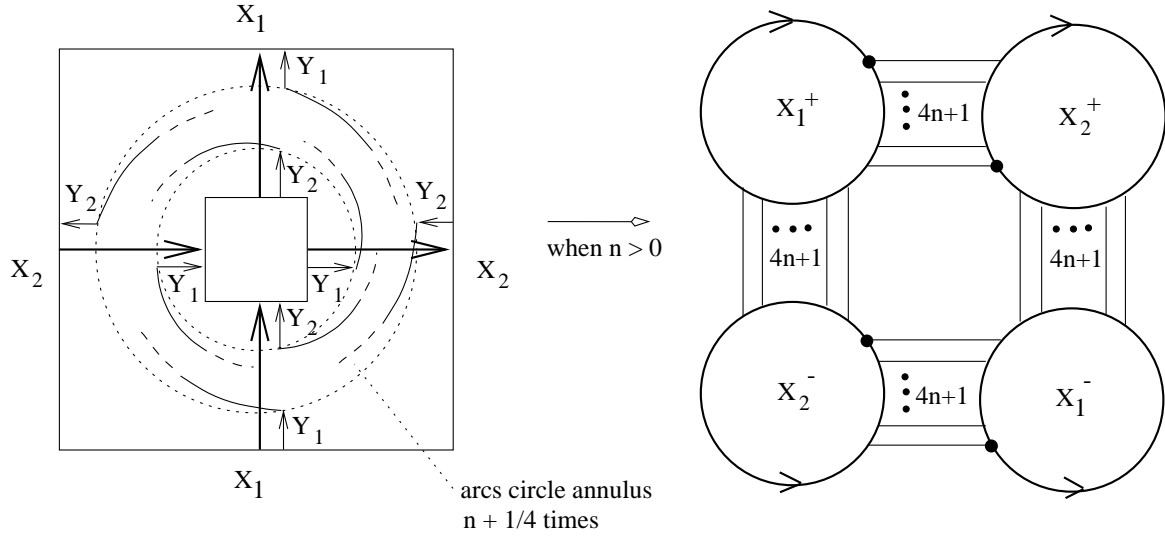
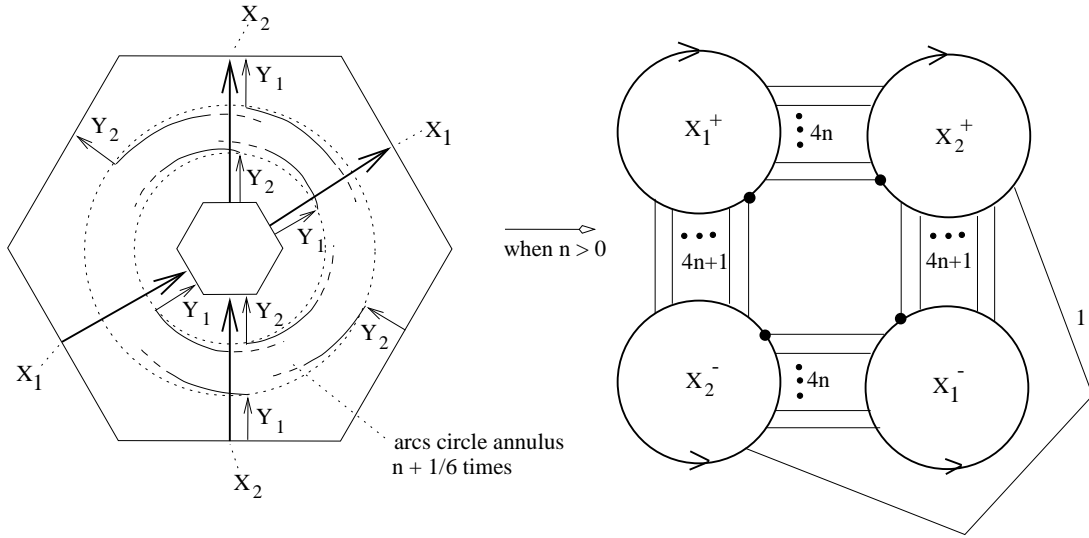


FIGURE 6A. Period three: $(g = 0, (1/3, 1/3, n/(3n \pm 1))); e = 1$.


 FIGURE 6B. Period four: $(g = 0, (1/2, 1/4, n/(4n \pm 1))); e = 1$).

 FIGURE 6C. Period three: $(g = 0, (1/2, 1/3, n/(6n \pm 1))); e = 1$).

The cases $k = 0$ all give lens spaces. In the cases $k = -1$, the horizontal splitting is also a vertical splitting (by, for example, [H2; Theorem 3.2]) and so is represented by a positive diagram and is excluded. The remaining possibilities follow very much as in the proof of case (1.2); with $k = 1, -2$, and $k = 2$ when the period p of ϕ is six requiring the extra arguments about crossings of two distinct stacks. This works except when $p = 6$ and $k = 1$. Here not every X -stack meets every Y -stack; so we cannot claim the properties of $J = a_1 \cup a_2 \cup b_1 \cup b_2$. This accounts for the final exclusion. \square

5. Remarks. There are some questions left open. With respect to Seifert manifolds (orientable with orientable base):

(1) We do not know whether the horizontal splittings of the manifolds

$$g = 0, m \geq 6 \text{ and even, invariants } = 1/2, \dots, 1/2, 1/3, e = m/2$$

or

$$g = 0, m = 3, \text{ invariants } = 1/2/1/3/1/7, e = 1$$

are represented by positive diagrams.

(2) We do not know whether the minimal genus vertical splittings with

$$g > 0 \text{ and } m \leq 2$$

are represented by positive diagrams.

More generally

(3) We do not know whether all vertical splittings (when $g > 0$) are represented by positive diagrams.

Outside the class of Seifert manifolds we know nothing.

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